
Algorithmic Framework for Model-based Reinforcement Learning with Theoretical Guarantees

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Abstract

While model-based reinforcement learning has empirically been shown to significantly reduce the sample complexity that hinders model-free RL, the theoretical understanding of such methods has been rather limited. In this paper, we introduce a novel algorithmic framework for designing and analyzing model-based RL algorithms with theoretical guarantees, and a practical algorithm Optimistic Lower Bounds Optimization (OLBO). In particular, we derive a theoretical guarantee of monotone improvement for model-based RL with our framework. We iteratively build a lower bound of the expected reward based on the estimated dynamical model and sample trajectories, and maximize it jointly over the policy and the model. Assuming the optimization in each iteration succeeds, the expected reward is guaranteed to improve. The framework also incorporates an optimism-driven perspective, and reveals the intrinsic measure for the model prediction error. Preliminary simulations demonstrate that our approach outperforms the standard baselines on continuous control benchmark tasks.

1. Introduction

In recent years reinforcement learning has achieved strong empirical success, including super-human performances on Atari games and Go (23; 31) and learning locomotion and manipulation skills in robotics (19; 29; 21). Many of these results are achieved by model-free reinforcement learning algorithms that often require a massive number of samples, and therefore their applications are mostly limited to simulated environments. Model-based reinforcement learning, in contrast, exploits the information from state observations

explicitly — by planning with a learned dynamical model — and is considered a promising approach to reduce the sample complexity. Indeed, empirical results (6; 7; 19; 25; 17; 27) have shown strong improvements in terms of sample efficiency.

Despite promising empirical findings, many of *theoretical* properties of model-based reinforcement are not well-understood. For example, how does the error of the estimated model affect the estimation of the value function and the planning? Can model-based RL algorithms be guaranteed to improve the policy monotonically and converge to a local maximum of the value function? How do we quantify the uncertainty in the dynamical models? Previous theoretical results (1; 32; 5; 35; 36) mostly focus on linear parametrizations of the value function, policy or dynamics, and thus may not be applicable to complex situations in deep reinforcement learning.

In this paper, we propose a novel algorithmic framework for model-based reinforcement learning with theoretical guarantees. We provide upper bound on how much the error can compound and divert the value of imaginary rollouts from their real value. With this, our algorithm builds a lower bound of the true value function from sample trajectories, and maximizes it over both the dynamical model and the policy. The real value function is guaranteed to monotonically increase (assuming the planning succeeds in each iteration.) To the best of our knowledge, this is the first theoretical guarantee of monotone improvement for model-based reinforcement learning. The framework also incorporates an optimism-driven perspective, and reveals the intrinsic measure of the model prediction error.

More concretely, the key idea of the paper is we can use an estimated model and sample trajectories to build a provable lower bound of the real value function V^π :

$$V^\pi \geq \widehat{V}^\pi - D^{\pi, \widehat{M}}. \quad (1.1)$$

Here \widehat{V}^π is the value function of the policy π on the estimated model \widehat{M} , and $D^{\pi, \widehat{M}}$ is a designed discrepancy bound D that captures the intrinsic difference between the estimated model \widehat{M} and the real dynamical model M^* .

Since discrepancy bound $D^{\pi, \widehat{M}}$ is *invariant* to the represen-

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tation of the state space it may lead to a better design of loss function for learning dynamical models. Under certain assumptions and simplification, our theory can recover the standard model-based RL algorithms but suggests that ℓ_2 or ℓ_1 norm loss is preferable compared the mean-squared error (MSE). As shown below, we indeed observe that ℓ_2 and ℓ_1 losses significantly outperform MSE baseline in continuous tasks in Mujoco (38).

We also justify our framework by showing that jointly optimizing policy and dynamical model yields better results compared to the standard scheme of tuning the model and policy separately. Readers may have realized that optimizing a robust lower bound is reminiscent of robust control and robust optimization. We remark that the vital distinction is that we optimistically and iteratively maximize the RHS of (1.1) jointly over the model \widehat{M} and the policy π , which compensates the conservatism from the lower bound.

Last but not the least, we remark that the most sophisticated theoretical results in our refined bound develop and utilize new mathematical tools that measure the difference between policies in χ^2 -divergence (instead of KL or TV). These tools may be of independent interests and used for better analysis of model-free reinforcement learning algorithms such as TRPO (28), PPO (30) and CPO (3).

1.1. Related Work

Model-based reinforcement learning are known to require fewer samples than model-free algorithms (7) and have been successfully applied to robotics in both simulation and in the real world (6; 24; 8) using dynamical models ranging from relatively simple models such as Gaussian process (6; 16), time-varying linear models (20; 22; 18; 39), mixture of Gaussians (14), to multi-layer neural networks (11; 25; 17; 37).

Prior work explores a variety of ways of combining model-free and model-based ideas to achieve the best of the two methods (34; 33). For example, learned models (20; 10; 13) are used to enrich the replay buffer in the model-free off-policy RL. The work of Pong et al. (37) proposes goal-conditioned value functions trained by model-free algorithms and use it for model-based controls. The work of Feinberg et al. (9) uses dynamical models to improve the estimation of the value functions in model-free algorithms.

The work (35; 36) analyzes the behavior of Dyna-like algorithms when both value function and dynamics are linear with TD(0). Abbeel et al.(2) shows such iterative approaches converge to a local optimum of expected reward, assuming the partial derivative of learned model is in the vicinity of the true models since optimization starts, which could be hard to satisfy. The work of Feinberg et al. (9) bounds the value discrepancy between the real and estimated models by

ℓ_2^2 error of the trajectories and use it to improve the value estimation in model-free RL. Our discrepancy bounds, in contrast, depend on the expected error of the model in one step and can be invariant to the state representation. Moreover, our algorithms enjoy convergence guarantees in the sense that the value under the unknown true dynamics is non-decreasing.

2. Notations and Preliminaries

We denote the state space by $\mathcal{S} \subset \mathbb{R}^d$, the action space by $\mathcal{A} \subset \mathbb{R}^\ell$. A policy $\pi(\cdot|\cdot)$ specifies the conditional distribution over the action space given a state. A dynamical model $M(\cdot|S, A)$ specifies the conditional distribution of the next state given the current state S and action A . We will use M^* globally to denote the unknown real dynamical model. Let \mathcal{M} denote a (parameterized) family of models that we are interested in. We use Π to denote the parameterized family of policies.

Let $S_0^{\pi, M}, \dots, S_t^{\pi, M}, \dots$ to denote the random variable of the states steps $0, \dots$, when we execute policy π on dynamic model M . We will omit the subscript when it's clear from the context. We use A_0, \dots, A_t, \dots for actions similarly. We often use τ to denote the random variables for the trajectory $(S_0, A_1, \dots, S_t, A_t, \dots)$. Let γ be the discount factor and $V^{\pi, M}$ be the value function:

$$V^{\pi, M}(s) = \mathbb{E}_{\substack{\forall t \geq 0, A_t \sim \pi(\cdot|S_t) \\ S_{t+1} \sim M(\cdot|S_t, A_t)}} \left[\sum_{t=0}^{\infty} \gamma^t R(S_t, A_t) \mid S_0 = s \right] \quad (2.1)$$

We define $V^{\pi, M} = \mathbb{E} [V^{\pi, M}(S_0)]$ as the expected reward-to-go at step 0 and our goal is to maximize the cumulative rewards V^{π, M^*} . For simplicity, throughout the paper, we set $\kappa = \gamma(1 - \gamma)^{-1}$ since it occurs frequently in our equations. Every policy π induces a distribution of states visited by policy π , as formally defined below.

Definition 2.1. For a policy π , define $\rho^{\pi, M}$ as the discounted distribution of the states visited by π on M . Let ρ^π be a shorthand for ρ^{π, M^*} and we omit the superscript M^* throughout the paper. Concretely, let $p_{S_t^\pi | S_0=s}$ be the distribution of $S_t^\pi \mid S_0^\pi = s$ and let $\rho^\pi = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t p_{S_t^\pi}$

3. Algorithmic Framework

As alluded in the introduction, towards optimizing V^{π, M^*} , our plan is to build a lower bound for V^{π, M^*} of the following type and optimize it iteratively:

$$V^{\pi, M^*} \geq V^{\pi, \widehat{M}} - D(\widehat{M}, \pi) \quad (3.1)$$

where $D(\widehat{M}, \pi) \in \mathbb{R}_{\geq 0}$ bounds from above the discrepancy between $V^{\pi, \widehat{M}}$ and V^{π, M^*} . Building such an optimizable discrepancy bound globally that holds for all \widehat{M} and π turns

out to be rather difficult, if not impossible. Instead, we shoot for establishing such a bound over the neighborhood of a reference policy π_{ref} .

$$V^{\pi, M^*} \geq V^{\pi, \widehat{M}} - D_{\pi_{\text{ref}}}(\widehat{M}, \pi), \quad \forall \pi \text{ s.t. } d(\pi, \pi_{\text{ref}}) \leq \delta \quad (\text{R1})$$

Here $d(\cdot, \cdot)$ is a function that measures the closeness of two policies (that will be chosen later in alignment with the choice of D .) We also note that the bound $D_{\pi_{\text{ref}}}(\widehat{M}, \pi)$ depends on the choice of neighborhood size δ but we omit a subscript for δ for simplicity. We will require our discrepancy bound to vanish when \widehat{M} is an accurate model:

$$\widehat{M} = M^* \implies D_{\pi_{\text{ref}}}(\widehat{M}, \pi) = 0, \quad \forall \pi, \pi_{\text{ref}} \quad (\text{R2})$$

The third requirement for the discrepancy bound D is that it can be estimated and optimized in the sense that

$$D_{\pi_{\text{ref}}}(\widehat{M}, \pi) \text{ is of the form } \mathbb{E}_{\tau \sim \pi_{\text{ref}}, M^*} [f(\widehat{M}, \pi, \tau)] \quad (\text{R3})$$

where f is a known differentiable function. We can estimate such kind of discrepancy bounds for *every* π in the neighborhood of π_{ref} by sampling empirical trajectories $\tau^{(1)}, \dots, \tau^{(n)}$ *once* from executing policy π_{ref} on the real environment M^* and compute the average of $f(\widehat{M}, \pi, \tau^{(i)})$'s. Note that we insist the expectation cannot be over the randomness of trajectories from π on M^* , because then we have to re-sample trajectories for every possible π encountered.

For example, one of the valid discrepancy bounds (under some assumptions) that we will prove in Section 4 is the error of the prediction of \widehat{M} on the trajectories from π_{ref} :

$$D_{\pi_{\text{ref}}}(\widehat{M}, \pi) = L \cdot \mathbb{E}_{S_0, \dots, S_t, \tau \sim \pi_{\text{ref}}, M^*} [\|\widehat{M}(S_t) - S_{t+1}\|] \quad (3.2)$$

Suppose we can establish such an discrepancy bound D (and the distance function d) with properties (R1), (R2), and (R3), — which will be the main focus of Section 4 —, then we can devise the following algorithmic framework as shown in Algorithm 1. We iteratively optimize the lower bound over the policy π_{k+1} and the model M_{k+1} , subject to the constraint that the policy is not very far from the reference policy π_k obtained in the previous iteration. For simplicity, we only state the population version with the exact computation of $D_{\pi_{\text{ref}}}(\widehat{M}, \pi)$.

We then show formally that the expected reward in the real environment is non-decreasing under the assumption that the real dynamics belongs to our parameterized family \mathcal{M} .

Theorem 3.1. *Suppose that $M^* \in \mathcal{M}$, that D and d satisfy equation (R1) and (R2), and the optimization problem in*

Algorithm 1 General Algorithmic Framework

Inputs: Initial policy π_0 . Discrepancy bound D and distance function d that satisfy equation (R1) and (R2).

For $k = 0$ to T :

$$\pi_{k+1}, M_{k+1} = \underset{\pi \in \Pi, M \in \mathcal{M}}{\operatorname{argmax}} V^{\pi, M} - D_{\pi_k}(M, \pi) \quad (3.3)$$

$$\text{s.t. } d(\pi, \pi_k) \leq \delta \quad (3.4)$$

equation (3.3) is solvable at each iteration. Then, Algorithm 1 produces a sequence of policies π_0, \dots, π_T with monotonically increasing values:

$$V^{\pi_0, M^*} \leq V^{\pi_1, M^*} \leq \dots \leq V^{\pi_T, M^*} \quad (3.5)$$

Moreover, as $k \rightarrow \infty$, the value V^{π_k, M^*} converges to some $V^{\bar{\pi}, M^*}$, where $\bar{\pi}$ is a local minimum of V^{π, M^*} in domain Π .

Proof. Since D and d satisfy equation (R1), we have that

$$V^{\pi_{k+1}, M^*} \geq V^{\pi_{k+1}, M_{k+1}} - D_{\pi_k}(M_{k+1}, \pi_{k+1})$$

By the definition that π_{k+1} and M_{k+1} are the optimizers of equation (3.3), we have that

$$V^{\pi_{k+1}, M_{k+1}} - D_{\pi_k}(M_{k+1}, \pi_{k+1}) \geq V^{\pi_k, M^*} - D_{\pi_k}(M^*, \pi_k) \quad (3.6)$$

$$= V^{\pi_k, M^*} \quad (\text{by equation R2})$$

Combing the two equations above we complete the proof of equation (3.5).

For the second part of the theorem, by compactness, we have that a subsequence of π_k converges to some $\bar{\pi}$. By the monotonicity we have $V^{\pi_k, M^*} \leq V^{\bar{\pi}, M^*}$ for every $k \geq 0$. For the sake of contradiction, we assume $\bar{\pi}$ is a not a local maximum, then in the neighborhood of $\bar{\pi}$ there exists π' such that $V^{\pi', M^*} > V^{\bar{\pi}, M^*}$ and $d(\bar{\pi}, \pi') < \delta/2$. Let t be such that π_t is in the $\delta/2$ -neighborhood of $\bar{\pi}$. Then we see that (π', M^*) is a better solution than (π_{t+1}, M_{t+1}) for the optimization in iteration t because $V^{\pi', M^*} > V^{\bar{\pi}, M^*} \geq V^{\pi_{t+1}, M^*} \geq V^{\pi_{t+1}, M_{t+1}} - D_{\pi_t}(M_{t+1}, \pi_t)$. (Here the last inequality uses equation (R1).) This contradicts the assumption that $\bar{\pi}$ is a local maximum. Therefore $\bar{\pi}$ is a local maximum and we complete the proof. \square

4. Discrepancy Bounds Design

In this section, we design discrepancy bounds that can provably satisfy the requirements (R1), (R2), and (R3). We design increasingly stronger discrepancy bounds from Section 4.1 to Section 4.2. However, we leave detailed proof and explanation to supp.

4.1. Norm-based prediction error Bounds

In this subsection, we derive the discrepancy bound D of the form $\|\widehat{M}(S, A) - M^*(S, A)\|$ averaged over the observed state-action pair (S, A) under certain conditions on the dynamical model \widehat{M} . This suggests that we should not use the mean-squared error for learning the model, and instead, we should use the norm itself as the metric. In section 5, we will demonstrate that the ℓ_2 norm error consistently outperforms the square of ℓ_2 norm. Through the derivation, we will also introduce a telescoping lemma, which serves as the main building block towards other finer discrepancy bounds.

In this subsection, we make the (strong) assumption that the imaginary value function $V^{\pi, \widehat{M}}$ is L -Lipschitz w.r.t to some norm $\|\cdot\|$ in the sense that

$$\forall s, s' \in \mathcal{S}, |V^{\pi, \widehat{M}}(s) - V^{\pi, \widehat{M}}(s')| \leq L \cdot \|s - s'\| \quad (4.1)$$

We start off with a lemma showing that the expected prediction error is an upper bound of the discrepancy between the real and imaginary values.

Lemma 4.1. *Suppose $V^{\pi, \widehat{M}}$ is L -Lipschitz (in the sense of (4.1)). Recall $\kappa = \gamma(1 - \gamma)^{-1}$. Then, we have*

$$|V^{\pi, \widehat{M}} - V^{\pi, M^*}| \leq \kappa L \mathbb{E}_{\substack{S \sim \rho^\pi \\ A \sim \pi(\cdot|S)}} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right] \quad (4.2)$$

However, in RHS in equation 4.2 cannot serve as a discrepancy bound because it doesn't satisfy the requirement (R3). We use the expected KL divergence between two π and π_{ref} to define the neighborhood:

$$d^{\text{KL}}(\pi, \pi_{\text{ref}}) = \mathbb{E}_{S \sim \rho^\pi} \left[KL(\pi(\cdot|S), \pi_{\text{ref}}(\cdot|S))^{1/2} \right] \quad (4.3)$$

Proposition 4.2. *In the same setting of Proposition 4.1, assume in addition that π is close to a reference policy π_{ref} in the sense that $d^{\text{KL}}(\pi, \pi_{\text{ref}}) \leq \delta$, and the states in \mathcal{S} are uniformly bounded in the sense that $\|s\| \leq B, \forall s \in \mathcal{S}$. Then,*

$$|V^{\pi, \widehat{M}} - V^{\pi, M^*}| \leq \kappa L \mathbb{E}_{\substack{S \sim \rho^{\pi_{\text{ref}}} \\ A \sim \pi(\cdot|S)}} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right] + 2\kappa^2 \delta B \quad (4.4)$$

In a benign scenario, the second term in the RHS of equation (4.4) should be dominated by the first term when the neighborhood size δ is sufficiently small.

$$G^{\pi, \widehat{M}}(s, a) = V^{\pi, \widehat{M}}(\widehat{M}(s, a)) - V^{\pi, \widehat{M}}(M^*(s, a)) \quad (4.5)$$

We give a telescoping lemma that decompose the discrepancy between $V^{\pi, \widehat{M}}$ and V^{π, M^*} into the expected single-step discrepancy G .

Lemma 4.3. *[Telescoping Lemma] Recall that $\kappa := \gamma(1 - \gamma)^{-1}$. For any policy π and dynamical models M, \widehat{M} , we have that*

$$V^{\pi, \widehat{M}} - V^{\pi, M} = \kappa \mathbb{E}_{\substack{S \sim \rho^{\pi, M} \\ A \sim \pi(\cdot|S)}} \left[G^{\pi, \widehat{M}}(S, A) \right] \quad (4.6)$$

We defer all the details to Supp. Materials.

4.2. Intrinsic error of the prediction

As alluded in the previous subsection, the prediction error based bounds doesn't apply when the value functions are not Lipschitz.

We propose the following discrepancy bound towards addressing the limitations. Recall the definition of $G^{\pi, \widehat{M}}(s, a) = V^{\pi, \widehat{M}}(\widehat{M}(s, a)) - V^{\pi, \widehat{M}}(M^*(s, a))$ which measures the difference between $\widehat{M}(s, a)$ and $M^*(s, a)$ according to their imaginary rewards. We construct a discrepancy bounds using the absolute value of G . Let's define ε_1 and ε_{max} as the average of $|G^{\pi, \widehat{M}}|$ and its maximum.

$$\varepsilon_1 = \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[|G^{\pi, \widehat{M}}(S)| \right] \quad \text{and} \quad \varepsilon_{\text{max}} = \max_S |G^{\pi, \widehat{M}}(S)| \quad (4.7)$$

We will show that the following discrepancy bound $D_{\pi_{\text{ref}}}^G(\widehat{M}, \pi)$ satisfies the property (R1), (R2).

$$D_{\pi_{\text{ref}}}^G(\widehat{M}, \pi) = \kappa \cdot \varepsilon_1 + \kappa^2 \delta \varepsilon_{\text{max}} \quad (4.8)$$

Proposition 4.4. *Let d^{KL} and D^G be defined as in equation (4.3) and (4.8). Then the choice $d = d^{\text{KL}}$ and $D = D^G$ satisfies the basic requirements (equation (R1) and (R2)).*

The proof follows from the telescoping lemma (Lemma 4.3) and is deferred to Supp Material. A refined bound is also deferred to Supp Material.

5. Experiments

We provide proof-of-concepts evaluations for the theory developed in the previous sections. We first demonstrate that our proposed algorithm, Optimistic Lower Bound Optimization (OLBO), which uses the norm-based discrepancy bound developed in Section 4.1 in the Framework 1, outperforms (in sample complexity) standard model-based RL baselines in swimmer and half-cheetach benchmark in mujoco. Second, we apply the discrepancy bound in Section ?? and show that it helps the performance in an artificial noisy half-cheetach environment, which indicates the discrepancy

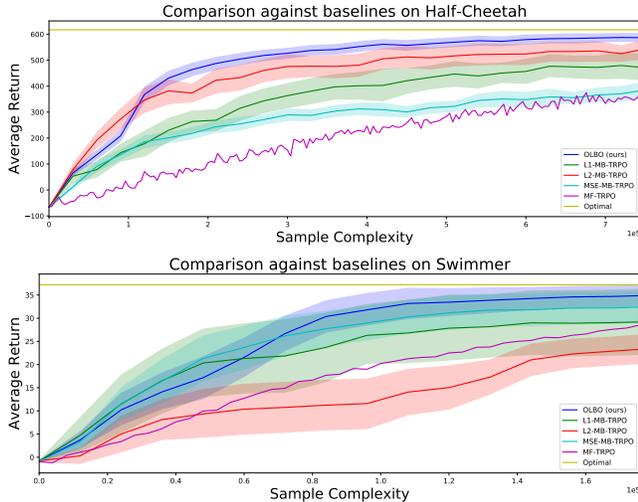


Figure 1: Performance on the benchmark Mujoco tasks. The x-axis shows the sample complexity (in the number of steps). The y-axis is the average return. Solid lines are average values over 5 random seeds for model-based method. Model-free learning curve is shown for reference. The initial $9e4$ steps of OLBO uses regular L1-MB-TRPO.

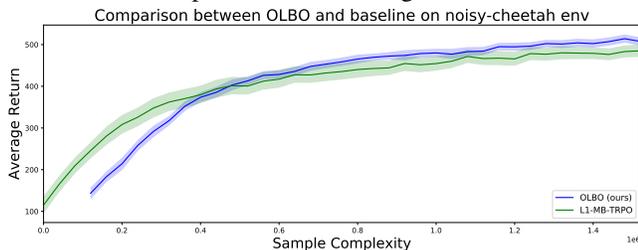


Figure 2: Performance on noisy-cheetah environment. The discrepancy loss helps the agent to focus more on the useful states from noisy redundant states. All the experiments are run on 4 random seeds.

bounds that are more robust to the change of state representations and thus may serve as a better learning objective for the dynamical model. Environment specs are in Appendix E.2.

Baselines The baselines include MSE-MB-TRPO, L1-MB-TRPO and L2-MB-TRPO, where the algorithms iteratively: a) learn the dynamical model with loss function MSE, L1 and L2 norm respectively, using the existing trajectories, b) policy optimization with the estimated dynamical model using TRPO (28), and c) collect more samples from the current policy. We call each ‘model-TRPO-data’ iteration a *stage* for simplicity and convenience.

We note that the standard algorithms (25; 17) use mean-squared error (MSE) as the loss function. Our theory in Section 4.1 indicates that the norm (instead of the square of the norm) is a better learning objective and therefore we include L1-MB-TRPO and L2-MB-TRPO as baseline

for completeness. Indeed, using the norm (either L1 or L2) as learning objective improves the performance. We perform these three simple but robust baselines with similar hyperparameter settings as detailed in Appendix.

OLBO on the benchmark We implement OLBO following the method in Sec.E.1. In each stage we optimize lower bound (3.3) (with discrepancy bounds being L1 as justified in Section 4.1) with respect to the model and the policy and collect the trajectories from the current policy. To optimize the lower bound (3.3), we alternate between maximizing the $V^{\pi, M}$ term using TRPO w.r.t the model and the policy (sing the reduction method in Section E.1) and minimizing the term $D(M, \pi)$ with respect to the model stochastically.¹ Empirically we found the constraint for the policy is not necessary and therefore drop it in the implementation.

We warm-start our algorithm after running 3 stages of L1-MB-TRPO baseline (in which $9e4$ steps of real rollouts are used). In Fig. 1, we find that all the model-based methods are more data efficient in the early stage of training compared to model-free TRPO. Only OLBO achieves near optimal performance with comparably small amount of data. We believe that the benefits come from two aspects of our algorithm: 1) The optimistic-driven approach provide the estimated model the flexibility to explore better possible model parameter while keeping it not far from the original one by regularizing it with a L1 loss. 2) The parameter of the estimated model and the policy network are jointly optimized so that the policy will have lower chance to overfit a fixed estimated model.

We share the same hyperparamters as baselines. But we have three more hyperparameters in OLBO. We jointly train model and policy in an iterative manner. We choose to run the model learning for 40 steps and then TRPO optimization for 10 steps. And will run TRPO optimization for a total of 800 steps.

Discrepancy Loss on Noisy Environment We test the proposed discrepancy loss (4.8) on an artificial cheetah environment to test whether our algorithm is more robust to different state representation. We enlarge the state space by 10 coordinates and add noise in them. We perform the L1-MB-TRPO baseline and train a critic network following the setting in DDPG (21) to estimate $V^{\pi, \hat{M}}$ that appear in the discrepancy bound. After 3 stages, we fix the critic network parameters and add the discrepancy loss (4.8) to the L1 objective with a coefficient $1e-3$. We find that adding this term will constantly improve the learned model and thus achieve higher rewards as shown in Figure 2. In practice, the critic learning is unstable especially with the estimated model. We slowdown the soft update rate 100 times. See Appendix E.2 for more implementation details.

¹Note that the norm discrepancy bound doesn’t depend on the policy.

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A. Proof of Lemma 4.3

Proof of Lemma 4.3. Let W_j be the cumulative reward when we use dynamical model M for j steps and then \widehat{M} for the rest of the steps, that is,

$$W_j = \mathbb{E}_{\substack{\forall t \geq 0, A_t \sim \pi(\cdot | S_t) \\ \forall j > t \geq 0, S_{t+1} \sim M(\cdot | S_t, A_t) \\ \forall t \geq j, S_{t+1} \sim \widehat{M}(\cdot | S_t, A_t)}} \left[\sum_{t=0}^{\infty} \gamma^t R(S_t, A_t) \mid S_0 = s \right] \quad (\text{A.1})$$

By definition, we have that $W_\infty = V^{\pi, M}(s)$ and $W_0 = V^{\pi, \widehat{M}}(s)$. Then, we decompose the target into a telescoping sum,

$$V^{\pi, M}(s) - V^{\pi, \widehat{M}}(s) = \sum_{j=0}^{\infty} (W_{j+1} - W_j) \quad (\text{A.2})$$

Now we re-write each of the summands $W_{j+1} - W_j$. Comparing the trajectory distribution in the definition of W_{j+1} and W_j , we see that they only differ in the dynamical model applied in j -th step. Concretely, W_j and W_{j+1} can be rewritten as $W_j = R + \mathbb{E}_{S_j, A_j \sim \pi, M} \left[\mathbb{E}_{\hat{S}_{j+1} \sim \widehat{M}(\cdot | S_j, A_j)} \left[\gamma^{j+1} V^{\pi, \widehat{M}}(\hat{S}_{j+1}) \right] \right]$ and $W_{j+1} = R + \mathbb{E}_{S_j, A_j \sim \pi, M^*} \left[\mathbb{E}_{S_{j+1} \sim M(\cdot | S_j, A_j)} \left[\gamma^{j+1} V^{\pi, \widehat{M}}(S_{j+1}) \right] \right]$ where R denotes the reward from the first j steps from policy π and model M^* . Canceling the shared term in the two equations above, we get

$$\begin{aligned} & W_{j+1} - W_j \\ &= \gamma^{j+1} \mathbb{E}_{S_j, A_j \sim \pi, M} \left[\mathbb{E}_{\substack{\hat{S}_{j+1} \sim \widehat{M}(\cdot | S_j, A_j) \\ S_{j+1} \sim M(\cdot | S_j, A_j)}} \left[V^{\pi, \widehat{M}}(S_{j+1}) - V^{\pi, \widehat{M}}(\hat{S}_{j+1}) \right] \right] \end{aligned} \quad (\text{A.3})$$

Combining the equation above with equation (A.2) concludes that

$$\begin{aligned} & V^{\pi, M} - V^{\pi, \widehat{M}} \\ &= \frac{\gamma}{1-\gamma} \mathbb{E}_{\substack{S \sim \rho^\pi, \\ A \sim \pi(S)}} \left[V^{\pi, \widehat{M}}(M(S, A)) - V^{\pi, \widehat{M}}(\widehat{M}(S, A)) \right] \end{aligned} \quad (\text{A.4})$$

□

B. Missing Proofs in Section 4

Proof of Proposition 4.4. By Lemma 4.3 and triangle inequality, we have that

$$\begin{aligned} \frac{1-\gamma}{\gamma} \left| V^{\pi, M} - V^{\pi, \widehat{M}} \right| &\leq \mathbb{E}_{S \sim \rho^\pi} \left[\left| G^{\pi, \widehat{M}}(S) \right| \right] \\ &\quad (\text{triangle inequality}) \\ &\leq \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[\left| G^{\pi, \widehat{M}}(S) \right| \right] \\ &\quad + \left| \rho^\pi - \rho^{\pi_{\text{ref}}} \right|_1 \cdot \max_S \left| G^{\pi, \widehat{M}}(S) \right| \\ &\quad (\text{Holder inequality}) \end{aligned} \quad (\text{B.1})$$

By Corollary D.7 we have that $\left| \rho^\pi - \rho^{\pi_{\text{ref}}} \right|_1 \leq \frac{\gamma}{1-\gamma} \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[KL(\pi(S), \pi_{\text{ref}}(S))^{1/2} \mid S \right] = \frac{\delta\gamma}{1-\gamma}$. Combining this with the equation above, we complete the proof. □

Proof of Proposition ??. Let μ be the distribution of the initial state S_0 , and let P' and P be the state-to-state transition kernel under policy π and π_{ref} . Let $\bar{G} = (1-\gamma) \sum_{k=0}^{\infty} \gamma^k P^k$ and $\bar{G}' = (1-\gamma) \sum_{k=0}^{\infty} \gamma^k P'^k$. Under these notations, we can re-write $\rho^{\pi_{\text{ref}}} = \bar{G}\mu$ and $\rho^\pi = \bar{G}'\mu$. Moreover, we observe that $\beta^{\pi_{\text{ref}}} = \bar{G}P\bar{G}\mu$.

Let $\delta_1 = (1-\gamma)\chi_{\bar{G}\mu}^2(P', P)^{1/2}$ and $\delta_2 = (1-\gamma)\chi_{\bar{G}P\bar{G}\mu}^2(P', P)^{1/2}$ by the χ^2 divergence between P' and P , measured with respect to distributions $\bar{G}\mu = \rho^{\pi_{\text{ref}}}$ and $\bar{G}P\bar{G}\mu = \beta^{\pi_{\text{ref}}}$. By Lemma C.1, we have that the χ^2 -divergence between the states can be bounded by the χ^2 -divergence between the actions in the sense that:

$$\begin{aligned} \chi_{\bar{G}\mu}^2(P', P)^{1/2} &= \chi_{\rho^{\pi_{\text{ref}}}}^2(P', P)^{1/2} \\ &\leq \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[\chi^2(\pi(\cdot | S), \pi_{\text{ref}}(\cdot | S)) \right] \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} \chi_{\bar{G}P\bar{G}\mu}^2(P', P)^{1/2} &= \chi_{\beta^{\pi_{\text{ref}}}}^2(P', P)^{1/2} \\ &\leq \mathbb{E}_{S \sim \beta^{\pi_{\text{ref}}}} \left[\chi^2(\pi(\cdot | S), \pi_{\text{ref}}(\cdot | S)) \right] \end{aligned} \quad (\text{B.3})$$

Therefore we obtain that $\delta_1 \leq (1-\gamma)\delta$, $\delta_2 \leq (1-\gamma)\delta$. Let $f(s) = G^{\pi, \widehat{M}}(s)$. We can control the difference between $\langle \rho^{\pi_{\text{ref}}}, f \rangle$ and $\langle \rho^\pi, f \rangle$ by

$$\left| \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[G^{\pi, \widehat{M}}(S) \right] - \mathbb{E}_{S \sim \rho^\pi} \left[G^{\pi, \widehat{M}}(S) \right] \right| \quad (\text{B.4})$$

$$= \left| \langle \rho^{\pi_{\text{ref}}}, f \rangle - \langle \rho^\pi, f \rangle \right| \quad (\text{B.5})$$

$$\leq (1-\gamma)^{-1} \left(\delta_1 \langle \bar{G}P\bar{G}\mu, f^2 \rangle^{1/2} + \delta_1 \delta_2^{1/2} \|f\|_\infty \right) \quad (\text{B.6})$$

$$= (1-\gamma)^{-1} \left(\delta_1 \varepsilon_2 + \delta_1 \delta_2^{1/2} \varepsilon_{\max} \right) \quad (\text{B.7})$$

$$\leq \delta \varepsilon_2 + (1-\gamma)^{-1/2} \delta_1 \delta_2^{1/2} \varepsilon_{\max} \quad (\text{B.8})$$

It follows that

$$\begin{aligned}
 & \left| V^{\pi, \widehat{M}} - V^{\pi, M} \right| \tag{B.9} \\
 & \leq \gamma(1-\gamma)^{-1} \left| \mathbb{E}_{S \sim \rho^\pi} \left[G^{\pi, \widehat{M}}(S) \right] \right| \quad (\text{by Lemma 4.3}) \\
 & \leq \gamma(1-\gamma)^{-1} \left(\left| \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[G^{\pi, \widehat{M}}(S) \right] \right| \right. \\
 & \quad \left. + \left| \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[G^{\pi, \widehat{M}}(S) - \mathbb{E}_{S \sim \rho^\pi} \left[G^{\pi, \widehat{M}}(S) \right] \right] \right| \right) \\
 & \leq \gamma(1-\gamma)^{-1} \left| \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} \left[G^{\pi, \widehat{M}}(S) \right] \right| \\
 & \quad + \gamma(1-\gamma)^{-1} \delta \varepsilon_2 + \gamma(1-\gamma)^{-3/2} \delta_1 \delta_2^{1/2} \varepsilon_{\max}
 \end{aligned}$$

□

Proof of Proposition 4.1 and 4.2. By definition of G and the Lipschitzness of $V^{\pi, \widehat{M}}$, we have that $|G^{\pi, \widehat{M}}(s, a)| \leq L|\widehat{M}(s, a) - M^*(s, a)|$. Then, by Lemma 4.3 and triangle inequality, we have that

$$\begin{aligned}
 |V^{\pi, \widehat{M}} - V^{\pi, M^*}| &= \kappa \cdot \left| \mathbb{E}_{\substack{S \sim \rho^{\pi, M} \\ A \sim \pi(\cdot|S)}} \left[G^{\pi, \widehat{M}}(S, A) \right] \right| \tag{B.10} \\
 & \leq \kappa \mathbb{E}_{\substack{S \sim \rho^{\pi, M} \\ A \sim \pi(\cdot|S)}} \left[|G^{\pi, \widehat{M}}(S, A)| \right] \\
 & \leq \kappa \mathbb{E}_{\substack{S \sim \rho^{\pi} \\ A \sim \pi(\cdot|S)}} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right]. \tag{B.11}
 \end{aligned}$$

Thus we proved Proposition 4.1. Note that for any distribution ρ and ρ' and function f , we have $\mathbb{E}_{S \sim \rho} f(S) = \mathbb{E}_{S \sim \rho'} f(S) + \langle \rho - \rho', f \rangle \leq \mathbb{E}_{S \sim \rho'} f(S) + \|\rho - \rho'\|_1 \|f\|_\infty$. Thus applying this inequality with $f(S) = \mathbb{E}_{A \sim \pi(\cdot|S)} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right]$, we obtain that

$$\begin{aligned}
 & \mathbb{E}_{\substack{S \sim \rho^\pi \\ A \sim \pi(\cdot|S)}} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right] \\
 & \leq \mathbb{E}_{\substack{S \sim \rho^{\pi_{\text{ref}}} \\ A \sim \pi(\cdot|S)}} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right] \\
 & \quad + \|\rho^{\pi_{\text{ref}}} - \rho\|_1 \max_S \mathbb{E}_{A \sim \pi(\cdot|S)} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right] \\
 & \leq \mathbb{E}_{\substack{S \sim \rho^{\pi_{\text{ref}}} \\ A \sim \pi(\cdot|S)}} \left[\|\widehat{M}(S, A) - M^*(S, A)\| \right] + 2\delta\kappa B
 \end{aligned}$$

where the last inequality uses the inequalities (see Corollary D.7) that $\|\rho^\pi - \rho^{\pi_{\text{ref}}}\|_1 \leq \frac{\gamma}{1-\gamma} \mathbb{E}_{S \sim \rho^{\pi_{\text{ref}}}} [KL(\pi(S), \pi_{\text{ref}}(S))^{1/2} | S] = \delta\kappa$ and that $\|\widehat{M}(S, A) - M^*(S, A)\| \leq 2B$. Combining (C.1) and (B.11) we complete the proof of Proposition 4.2. □

C. χ^2 -Divergence Based Inequalities

Lemma C.1. *Let S be a random variable over the domain \mathcal{S} . Let π and π' be two policies and and $A \sim \pi(\cdot | S)$ and $A' \sim \pi'(\cdot | S)$. Let $Y \sim M(\cdot | S, A)$ and $Y' \sim M(\cdot | S, A')$ be the random variables for the next states under two policies. Then,*

$$\mathbb{E} [\chi^2(Y|S, Y'|S)] \leq \mathbb{E} [\chi^2(A|S, A'|S)] \tag{C.1}$$

Proof. By definition, we have that $Y|S = s, A = a$ has the same density as $Y'|S = s, A' = a$ for any a and s . Therefore by Theorem D.4 (setting X, X', Y, Y' in Theorem D.4 by $A|S = s, A'|S = s, Y|S = s, Y'|S = s$ respectively), we have

$$\chi^2(Y|S = s, Y'|S = s) \leq \chi^2(A|S = s, A'|S = s) \tag{C.2}$$

Taking expectation over the randomness of S we complete the proof. □

C.1. Properties of Markov Processes

In this subsection, we consider bounded the difference of the distributions induced by two markov process starting from the same initial distributions μ . Let P, P' be two transition kernels. Let $G = \sum_{k=0}^{\infty} \gamma^k P^k$ and $\bar{G} = (1-\gamma)G$. Define G' and \bar{G}' similarly. Therefore we have that $\bar{G}\mu$ is the discounted distribution of states visited by the markov process starting from distribution μ . In other words, if μ is the distribution of S_0 , and P is the transition kernel induced by some policy π , then $\bar{G}\mu = \rho^\pi$.

First of all, let $\Delta = \gamma(P' - P)$ and we note that with simple algebraic manipulation,

$$\bar{G}' - \bar{G} = (1-\gamma)^{-1} \bar{G}' \Delta \bar{G} \tag{C.3}$$

Let f be some function. We will mostly interested in the difference between $\mathbb{E}_{S \sim \bar{G}\mu} [f]$ and $\mathbb{E}_{S \sim \bar{G}'\mu} [f]$, which can be rewritten as $\langle (\bar{G}' - \bar{G})\mu, f \rangle$. We will bound this quantity from above by some divergence measure between P' and P .

We start off with a simple lemma that controls the form $\langle p - q, f \rangle$ by the χ^2 divergence between p and q . With this lemma we can reduce our problem of bounding $\langle (\bar{G}' - \bar{G})\mu, f \rangle$ to characterizing the χ^2 divergence between $\bar{G}'\mu$ and $\bar{G}\mu$.

Lemma C.2. *Let p and q be probability distributions. Then we have*

$$\langle q - p, f \rangle^2 \leq \chi^2(q, p) \cdot \langle p, f^2 \rangle$$

Proof. By Cauchy-Schwartz inequality, we have

$$\begin{aligned} \langle q - p, f \rangle^2 &\leq \left(\int \frac{(q(x) - p(x))^2}{p(x)} dx \right) \left(\int p(x) f(x)^2 \right) \\ &= \chi^2(q, p) \cdot \langle p, f^2 \rangle \end{aligned} \quad (\text{C.4})$$

□

The following Lemma is a refinement of the lemma above. It deals with the distributions p and q with the special structure $p = WP'\mu$ and $q = WP\mu$.

Lemma C.3. *Let W, P', P be transition kernels and μ be a distribution. Then,*

$$\langle W(P' - P)\mu, f \rangle^2 \leq \chi_\mu^2(P', P) \langle WP\mu, f^2 \rangle \quad (\text{C.5})$$

where $\chi_\mu^2(P', P)$ is a divergence between transitions defined in Definition D.3.

Proof. By Lemma C.2 with $p = WP\mu$ and $q = WP'\mu$, we conclude that

$$\begin{aligned} \langle W(P' - P)\mu, f \rangle^2 &\leq \chi^2(q, p) \cdot \langle p, f^2 \rangle \\ &\leq \chi^2(WP'\mu, WP\mu) \langle WP\mu, f^2 \rangle \end{aligned} \quad (\text{C.6})$$

By Theorem D.4 and Theorem D.5 we have that $\chi^2(WP'\mu, WP\mu) \leq \chi^2(P'\mu, P\mu) \leq \chi_\mu^2(P', P)$, plugging this into the equation above we complete the proof. □

Now we are ready to state the main result of this subsection.

Lemma C.4. *Let $\bar{G}, \bar{G}', P', P, f$ as defined in the beginning of this section. Let $\delta_1 = (1 - \gamma)\chi_{\bar{G}\mu}^2(P', P)^{1/2}$ and $\delta_2 = (1 - \gamma)\chi_{\bar{G}P\bar{G}\mu}^2(P', P)^{1/2}$. Then,*

$$(1 - \gamma) |\langle \bar{G}'\mu, f \rangle - \langle \bar{G}\mu, f \rangle| \leq \delta_1 \|f\|_\infty \quad (\text{C.7})$$

$$(1 - \gamma) |\langle \bar{G}'\mu, f \rangle - \langle \bar{G}\mu, f \rangle| \leq \delta_1 \langle \bar{G}P\bar{G}\mu, f^2 \rangle^{1/2} + \delta_1 \delta_2^{1/2} \|f\|_\infty \quad (\text{C.8})$$

Proof. Recall by equation (C.3), we have $(\bar{G}' - \bar{G})\mu = \langle \bar{G}'\Delta\bar{G}\mu, f \rangle$. By Lemma C.3,

$$\langle \bar{G}'\Delta\bar{G}\mu, f \rangle^2 \leq \chi_{\bar{G}\mu}^2(P', P) \langle \bar{G}'P\bar{G}\mu, f^2 \rangle \quad (\text{C.9})$$

Using equation (C.3) again, we have

$$\langle \bar{G}'P\bar{G}\mu, f^2 \rangle = \langle \bar{G}P\bar{G}\mu, f^2 \rangle + \frac{1}{1 - \gamma} \langle \bar{G}'\Delta\bar{G}P\bar{G}\mu, f^2 \rangle \quad (\text{C.10})$$

By Lemma C.3 again, we have that

$$\langle \bar{G}'\Delta\bar{G}P\bar{G}\mu, f^2 \rangle \leq \chi_{\bar{G}P\bar{G}\mu}^2(P', P) \langle \bar{G}'P\bar{G}P\bar{G}\mu, f^4 \rangle \quad (\text{C.11})$$

By Holder inequality and the fact that $\|\bar{G}\|_{1 \rightarrow 1} = 1$, $\|\bar{G}'\|_{1 \rightarrow 1} = 1$ and $\|P\|_{1 \rightarrow 1} = 1$, we have

$$\langle \bar{G}'P\bar{G}P\bar{G}\mu, f^4 \rangle \leq \|\bar{G}'P\bar{G}P\bar{G}\mu\|_1 \|f^4\|_\infty \leq \|f\|_\infty^4 \quad (\text{C.12})$$

Then, combining equation (C.9), (C.10), (C.12), we have

$$\begin{aligned} &(1 - \gamma) |\langle \bar{G}'\Delta\bar{G}\mu, f \rangle| \quad (\text{C.13}) \\ &\leq \delta_1 \langle \bar{G}'P\bar{G}\mu, f^2 \rangle^{1/2} \quad (\text{by equation (C.9)}) \\ &\leq \delta_1 \langle \bar{G}P\bar{G}\mu, f^2 \rangle^{1/2} + \delta_1 \langle \bar{G}'\Delta\bar{G}P\bar{G}\mu, f^2 \rangle^{1/2} \\ &\quad (\text{by equation (C.10) and AM-GM}) \\ &\leq \delta_1 \langle \bar{G}P\bar{G}\mu, f^2 \rangle^{1/2} + \delta_1 \delta_2^{1/2} \|f\|_\infty \\ &\quad (\text{by equation (C.11) and (C.12)}) \end{aligned}$$

□

The following Lemma is a stronger extension of Lemma C.4, which can be of use to

Lemma C.5. *Let $\bar{G}, \bar{G}', P', P, f$ as defined in the beginning of this section. Let $d_k = (\bar{G}P)^k \bar{G}\mu$ and $\delta_k = (1 - \gamma)\chi_{d_{k-1}}^2(P', P)^{1/2}$, then we have that for any K ,*

$$\begin{aligned} &(1 - \gamma) |\langle \bar{G}'\mu, f \rangle - \langle \bar{G}\mu, f \rangle| \quad (\text{C.14}) \\ &\leq \delta_1 \langle d_1, f^2 \rangle^{-1/2} + \delta_1 \delta_2^{-1/2} \langle d_2, f^4 \rangle^{-1/4} + \\ &\quad + \delta_1 \dots \delta_{K-1}^{-2-K} \langle d_k, f^{2^k} \rangle^{2^{-k}} + \delta_1 \dots \delta_{K-1}^{2-K} \|f\|_\infty \end{aligned}$$

Proof. We first use induction to prove that:

$$\begin{aligned} &(1 - \gamma) |\langle (\bar{G}' - \bar{G})\mu, f \rangle| \\ &\leq \sum_{k=1}^K \left(\prod_{0 \leq s \leq k-1} \delta_{s+1}^{2^{-s}} \right) \langle d_k, f^{2^k} \rangle^{2^{-k}} \\ &\quad + \left(\prod_{0 \leq s \leq K-1} \delta_{s+1}^{2^{-s}} \right) \langle \bar{G}'\Delta(\bar{G}P)^K \bar{G}\mu, f^{2^K} \rangle^{2^{-K}} \end{aligned} \quad (\text{C.15})$$

By Lemma C.4, we got the case for $K = 1$. Assuming we have prove the case for K , then applying

$$\begin{aligned} &\langle \bar{G}'\Delta(\bar{G}P)^K \bar{G}\mu, f^{2^K} \rangle \\ &= \langle \bar{G}\Delta(\bar{G}P)^K \bar{G}\mu, f^{2^K} \rangle \\ &\quad + (1 - \gamma)^{-1} \langle \bar{G}\Delta(\bar{G}P)^{K+1} \bar{G}\mu, f^{2^K} \rangle \\ &\leq \langle \bar{G}\Delta(\bar{G}P)^K \bar{G}\mu, f^{2^K} \rangle \\ &\quad + (1 - \gamma)^{-1} \chi_{d_K}^2(P', P)^{1/2} \langle \bar{G}'\Delta(\bar{G}P)^{K+1} \bar{G}\mu, f^{2^{K+1}} \rangle^{1/2} \\ &\leq \langle \bar{G}\Delta(\bar{G}P)^K \bar{G}\mu, f^{2^K} \rangle \\ &\quad + \delta_{K+1} \langle \bar{G}'\Delta(\bar{G}P)^{K+1} \bar{G}\mu, f^{2^{K+1}} \rangle^{1/2} \end{aligned}$$

By Cauchy-Schwartz inequality, we obtain that

$$\langle \bar{G}' \Delta (\bar{G}P)^K \bar{G} \mu, f^{2^K} \rangle^{2^{-K}} \quad (\text{C.16})$$

$$\leq \langle \bar{G} \Delta (\bar{G}P)^K \bar{G} \mu, f^{2^K} \rangle^{2^{-K}} \quad (\text{C.17})$$

$$+ \delta_{K+1} \langle \bar{G}' \Delta (\bar{G}P)^{K+1} \bar{G} \mu, f^{2^{K+1}} \rangle^{2^{-K-1}}$$

Plugging the equation above into equation (C.15), we provide the induction hypothesis for the case with $K + 1$.

Now applying $\langle \bar{G}' \Delta (\bar{G}P)^K \bar{G} \mu, f^{2^K} \rangle^{2^{-K}} \leq \|f\|_\infty$ with equation (C.15) we complete the proof. \square

D. Toolbox

Definition D.1 (χ^2 distance, c.f. (26; 4)). The Neyman χ^2 distance between two distributions p and q is defined as

$$\chi^2(p, q) \triangleq \int \frac{(p(x) - q(x))^2}{q(x)} dx = \int \frac{p(x)^2}{q(x)} dx - 1 \quad (\text{D.1})$$

For notational simplicity, suppose two random variables X and Y has distributions p_X and p_Y , we often write $\chi^2(X, Y)$ as a simplification for $\chi^2(p_X, p_Y)$. \square

Theorem D.2 ((?)). The Kullback-Leibler (KL) divergence between two distributions p, q is bounded from above by the χ^2 distance:

$$KL(p, q) \leq \chi^2(p, q) \quad (\text{D.2})$$

Proof. Since log is a concave function, by Jensen inequality we have

$$\begin{aligned} KL(p, q) &= \int p(x) \log \frac{p(x)}{q(x)} dx \leq \log \int p(x) \cdot \frac{p(x)}{q(x)} dx \\ &= \log(\chi^2(p, q) + 1) \leq \chi^2(p, q) \end{aligned}$$

\square

Definition D.3 (χ^2 distance between transitions). Given two transition kernels P, P' . For any distribution μ , we define $\chi_\mu^2(P', P)$ as:

$$\chi_\mu^2(P', P) \triangleq \int \mu(x) \chi^2(P'(\cdot|X=x), P(\cdot|X=x)) dx \quad (\text{D.3})$$

Theorem D.4. Suppose random variables (X, Y) and (X', Y') satisfy that $p_{Y|X} = p_{Y'|X'}$. Then

$$\chi^2(Y, Y') \leq \chi^2(X, X') \quad (\text{D.4})$$

Or equivalently, for any transition kernel P and distribution μ, μ' , we have

$$\chi^2(P\mu, P\mu') \leq \chi^2(\mu, \mu') \quad (\text{D.5})$$

Proof. Denote $p_{Y|X}(y|x) = p_{Y'|X'}(y|x)$ by $p(y|x)$, and we rewrite p_X as p and $p_{X'}$ as p' . By Cauchy-Schwarz inequality, we have:

$$\begin{aligned} p_Y(y)^2 &= \left(\int p(y|x)p(x) dx \right)^2 \\ &\leq \left(\int p(y|x)p'(x) dx \right) \left(\int p(y|x) \frac{p(x)^2}{p'(x)} dx \right) \\ &= p_{Y'}(y) \left(\int p(y|x) \frac{p(x)^2}{p'(x)} dx \right) \end{aligned}$$

It follows that

$$\begin{aligned} \chi^2(Y, Y') &= \int \frac{p_Y(y)^2}{p_{Y'}(y)} dy - 1 \\ &\leq \int dy \int p(y|x) \frac{p(x)^2}{p'(x)} dx - 1 \\ &= \chi^2(X, X') \end{aligned}$$

\square

Theorem D.5. Let X, Y, Y' are three random variables. Then,

$$\chi^2(Y, Y') \leq \mathbb{E} [\chi^2(Y|X, Y'|X)] \quad (\text{D.6})$$

We note that the expectation on the right hand side is over the randomness of X .² As a direct corollary, we have for transition kernel P' and P and distribution μ ,

$$\chi^2(P'\mu, P\mu) \leq \chi_\mu^2(P', P) \quad (\text{D.7})$$

Proof. We denote $p_{Y'|X}(y|x)$ by $p'(y|x)$ and $p_{Y|X}(y|x)$ by $p(y|x)$, and let $p(x)$ be a simplification for $p_X(x)$. We have by Cauchy-Schwarz,

$$\frac{p_Y(y)^2}{p_{Y'}(y)} = \frac{(\int p(y|x)p(x) dx)^2}{\int p'(y|x)p(x) dx} \leq \int \frac{p(y|x)^2}{p'(y|x)} p(x) dx \quad (\text{D.8})$$

It follows that

$$\begin{aligned} \chi^2(Y, Y') &= \int \frac{p_Y(y)^2}{p_{Y'}(y)} dy - 1 \\ &\leq \int \frac{p(y|x)^2}{p'(y|x)} p(x) dx dy - 1 \\ &= \mathbb{E} [\chi^2(Y|X, Y'|X) | X] \end{aligned} \quad (\text{D.9})$$

\square

²Observe $\chi^2(Y|X, Y'|X)$ deterministically depends on X .

Claim D.6. Let μ be a distribution over the state space \mathcal{S} . Let P and P' be two transition kernels. $G = \sum_{k=0}^{\infty} (\gamma P)^k = (\text{Id} - \gamma P)^{-1}$ and $G' = \sum_{k=0}^{\infty} (\gamma P')^k = (\text{Id} - \gamma P')^{-1}$. Let $d = (1 - \gamma)G\mu$ and $d' = (1 - \gamma)G'\mu$ be the discounted distribution starting from μ induced by the transition kernels G and G' . Then,

$$|d - d'|_1 \leq \frac{1}{1 - \gamma} |\Delta d|_1 \quad (\text{D.10})$$

Moreover, let $\gamma(P' - P) = \Delta$. Then, we have

$$G' - G = \sum_{k=1}^{\infty} (G\Delta)^k G \quad (\text{D.11})$$

Proof. With algebraic manipulation, we obtain,

$$\begin{aligned} G' - G &= (\text{Id} - \gamma P')^{-1} ((\text{Id} - \gamma P) - (\text{Id} - \gamma P')) (\text{Id} - \gamma P)^{-1} \\ &= G' \Delta G \end{aligned} \quad (\text{D.12})$$

It follows that

$$\begin{aligned} |d - d'|_1 &= (1 - \gamma) |G' \Delta G \mu|_1 \leq |\Delta G \mu|_1 \\ &\quad (\text{since } (1 - \gamma) |G'|_{1 \rightarrow 1} \leq 1) \\ &= \frac{1}{1 - \gamma} |\Delta d|_1 \end{aligned}$$

Replacing G' in the RHS of the equation (D.12) by $G' = G + G' \Delta G$, and doing this recursively gives

$$G' - G = \sum_{k=1}^{\infty} (G\Delta)^k G$$

□

Corollary D.7. Let π and π' be two policies and let ρ^π be defined as in Definition 2.1. Then,

$$|\rho^\pi - \rho^{\pi'}|_1 \leq \frac{\gamma}{1 - \gamma} \mathbb{E}_{S \sim \rho^\pi} \left[KL(\pi(S), \pi'(S))^{1/2} \mid S \right] \quad (\text{D.13})$$

Proof. Let P and P' be the state-state transition matrix under policy π and π' and $\Delta = \gamma(P' - P)$. By Claim D.6, we have that

$$\begin{aligned} |\rho^\pi - \rho^{\pi'}|_1 &\leq \frac{1}{1 - \gamma} |\Delta \rho^\pi|_1 \\ &= \frac{\gamma}{1 - \gamma} \mathbb{E}_{S \sim \rho^\pi} \left[|p_{M^*(S, \pi(S))|S} - p_{M^*(S, \pi'(S))|S}|_1 \right] \\ &\leq \frac{\gamma}{1 - \gamma} \mathbb{E}_{S \sim \rho^\pi} \left[|p_{\pi(S)|S} - p_{\pi'(S)|S}|_1 \right] \\ &\leq \frac{\gamma}{1 - \gamma} \mathbb{E}_{S \sim \rho^\pi} \left[KL(\pi(S), \pi'(S))^{1/2} \mid S \right] \end{aligned} \quad (\text{D.14})$$

□

E. Implementation Details

E.1. Implementing Optimism-driven Approach by Reduction to Standard RL

In Algorithm 1, we propose to maximize $V - D$ over the policy and the dynamical model. We have discussed the optimization of discrepancy bounds when define them. In this section we show that maximizing $V^{\pi, M}$ over the policy π and the model M can be reduced to a standard RL problem and therefore we can call any model-free RL algorithms.

The reduction works by designing a new environment \widetilde{M} and new policy $\widetilde{\pi}$ so that optimizing $\widetilde{\pi}$ only with environment \widetilde{M} is equivalent to optimizing (π, M) . Concretely, we enlarge the action space to be $\widetilde{\mathcal{A}} = \mathcal{A} \times \mathcal{S}$. The policy $\widetilde{\pi}$ returns a concatenation of the real action produced by π and the next state, and the dynamical model \widetilde{M} just read off the next state from action $\widetilde{\pi}(S)$:

$$\forall s \in \mathcal{S}, \tilde{a} \in \mathcal{A} \times \mathcal{S}, \quad (\text{E.1})$$

$$\widetilde{\pi}(s) \triangleq (\pi(s), M(s, \pi(s))) \text{ and } \widetilde{M}(s, \tilde{a}) \triangleq \tilde{a}_S \quad (\text{E.2})$$

Here \tilde{a}_S denotes the restriction of a into the second set of coordinates w.r.t. the space \mathcal{S} .³

We have that $(\widetilde{M}, \widetilde{\pi})$ is equivalent to (M, π) in terms of the distributions of the states, and \widetilde{M} is fixed dynamics. Therefore, optimizing $\widetilde{\pi}$ is equivalent to optimizing (M, π) . Moreover, we note that such a translation preserves most, if not all, properties of the parameterization of π and M : if π and M are deterministic and differentiable, then so is $\widetilde{\pi}$; if stochastic π and M can be efficiently sampled, or are reparameterizable (15), so is $\widetilde{\pi}$; if the density of π and M can be evaluated, then so is $\widetilde{\pi}(\tilde{a}|s) = \pi(\tilde{a}_A|s) \cdot M(\tilde{a}_S|s, \tilde{a}_A)$;

E.2. Environment Specs

We adopt Half-Cheetah and Swimmer environments from Mujoco simulator as our testing environment. The observation space is slightly modified by removing redundant dimensions. The action space is normalized to the space from -1 to 1. We use the standard reward as in rllab. We set 200 as the maximum step number in each rollout. For the noisy-cheetah environment, we concatenate 10 more entries of unit Gaussian variable to the original states. This will make the environment harder to learn in practice.

E.3. Architecture

For all the experiments, we use a two layer fully connected network with 500 hidden units each layer to fit the dynamics of environments. We use another fully connected two-layer network with 32 hidden units per layer as policy network. To

³When π is stochastic, the two occurrences of $\pi(s)$ in equation (E.2) are defined to use the same randomness.

Table 1: Reward function definition in Swimmer and Half-cheetah environments

| | Reward r_t |
|--------------|--|
| Swimmer | $s_t^{xvel} - 0.5 \left\ \frac{a}{50} \right\ _2^2$ |
| Half-cheetah | $s_t^{xvel} - 0.05 \left\ \frac{a}{1} \right\ _2^2$ |

approximate the critic function, we use a two-layer network with 100 hidden units per layer. All the activation layers for model learning are using ReLU and for policy network tanh.

E.4. Data Preprocessing and Hyper-parameter Selection

E.4.1. DATA COLLECTION

We first define each *stage* means a full pipeline of data collection, model training and policy optimization. We populate the dataset with 60 rollouts in Swimmer and 150 rollouts in cheetah each *stage* that resulted from the execution of parameterized actions a_t from a randomly initialized exploratory policy network. Each rollout started from a selected starting state s_0 with small gaussian noise $\sim \mathcal{N}(0, 0.001)$. During data collection, we use the Ornstein-Uhlenbeck noise with $\sigma = 0.3$ and $\theta = 0.15$ to get an exploratory policy for more diverse trajectories. We set the upper bound scale for the dataset 5 times the amount collected in each *stage* and will drop the oldest ones when the dataset is full.

E.4.2. IMAGINARY MODEL LEARNING

We use Adam optimizer with $1e-3$ as learning rate to learn the model. We normalize the input data for model learning with empirical mean and variance computed in the initial *stage*. The model outputs a normalized difference between previous input state and the next state. The batchsize is 512 state and action pairs every step. We train this network $40/\min(n,5)$ epochs where n is the number of *stage* we are at. In each epoch, the network goes over the whole dataset once.

E.4.3. TRPO HYPERPARAMETERS

For TRPO we use the version provided by rllab and use a batch size of 4000 every iteration and step size 0.01 for conjugate gradient optimization. Other hyperparameters stay the same as the online version. In each stage we train it for 100 steps. Note that more training steps will make trpo overfit the imaginary model heavily for baselines.